Implementing Arrow-Debreu Equilibria by Trading Infinitely-Lived Securities. *

Kevin X.D. Huang and Jan Werner

Department of Economics, University of Minnesota

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1. Introduction

Equilibrium model of an dynamic economy extending over an infinite sequence of dates plays an important role in modern economic theory. The basic equilibrium concept in such model is the Arrow-Debreu (or Walrasian) competitive equilibrium. In an Arrow-Debreu equilibrium it is assumed that agents can simultaneously trade arbitrary consumption plans for the entire infinite and state-contingent future.

In most applications of a model a dynamic economy a market structure is used which is different from the Arrow-Debreu market structure. Instead of trading arbitrary consumption plans in simultaneous markets agents trade securities in sequential markets at every date, in every event. The importance of an Arrow-Debreu equilibrium rests on the possibility of *implementing* equilibrium allocations by trading suitable securities in sequential markets.

The idea of implementing an Arrow-Debreu equilibrium allocation by trading securities takes its origin in the classical paper by Arrow (1964). Arrow proved that every Arrow-Debreu equilibrium allocation in a two-period economy can be implemented by trading in spot commodity markets at every date and complete security markets in the first period. The implementation in the Arrow's model is *exact*: the sets of equilibrium allocations in the two market structures are exactly the same. Duffie and Huang (1985) extended the result of Arrow to an economy with continuous time (with finite time-horizon).

In this paper we investigate the implementation of Arrow-Debreu equilibrium allocations in an infinite-time economy by sequential trading of infinitely-lived securities. Wright (1987) studied a similar question with one-period-lived securities.

The crucial aspect of implementation in infinite-time security markets is the choice of feasibility constraints on agents' portfolio strategies. On the one hand, a feasibility constraint is necessary for without it agents would be able to borrow in security markets and roll over the debt without ever repaying it (Ponzi scheme). On the other hand, a constraint cannot be too "tight" for it could prevent agents from using portfolio strategies that generate wealth transfers necessary to achieve a consumption plan of an Arrow-Debreu equilibrium. Wright (1987) employs the *wealth constraint* which says that a consumer cannot borrow more than the present value of her future endowments. He proved that exact implementation holds with one-period-lived securities: the set of Arrow-Debreu equilibrium allocations and the set of equilibrium allocations in sequential markets are the same.

The difficulty in extending the implementation to infinitely-lived securities lies in the possibility of price bubbles in sequential markets. As pointed out by Huang and Werner (1999), the wealth constraint gives rise to sequential equilibria with price bubbles on securities that are in zero supply. (It follows from Santos and Woodford (1997) that there cannot be price bubbles on short-lived securities or on long-lived securities in zero supply.) We prove that if all securities are in strictly positive supply and security markets are complete, then the exact implementation of Arrow-Debreu equilibria obtains. If some securities are in zero supply, then Arrow-Debreu equilibria correspond to sequential equilibria with no price bubbles. Usually, there are also sequential equilibria with nonzero price bubbles. We show that these equilibria correspond to Arrow-Debreu equilibria with income transfers, where the transfers are given by the value of price bubbles on agents initial portfolio holdings.

We consider a second portfolio feasibility constraint: the *bounded borrowing constraint*. Under this constraint the value of a consumer's portfolio normalized with respect to some reference portfolio has to be bounded from below. This portfolio constraint has a remarkable feature that there cannot be a price bubble in an sequential equilibrium independently of whether the supply of the securities is strictly positive or zero. We prove that exact implementation of Arrow-Debreu equilibria holds under the bounded borrowing constraint.

The equlibrium theory in infinite-time economies (see Aliprantis, Brown and Burkinshaw (1987)) has produced several versions of the concept of an Arrow-Debreu equilibrium distinguished primarily by a specification of a consumption space and a space of prices. It is worth pointing out that Arrow-Debreu equilibria that, according to our results, can be implemented by sequential trading of securities have always countably additive prices. The concept of an equilibrium we employ is closely related to that of Peleg and Yaari (1971).

The paper is organized as follows: In section 2 we provide a specification of time and uncertainty. In section 3 we introduce the notion of Arrow-Debreu equilibrium and in section 4 we define a sequential equilibrium in security markets. We assume that there is a finite number of infinitely-lived agents and a a finite number of infinitely-lived securities available for trade at every date. In section 5 we state and prove our basic implementation results.

2. Time and Uncertainty

Time is discrete with infinite horizon and indexed by $t = 0, 1 \ldots$. Uncertainty is described by a set $\mathcal{S}$ of states of the world and an increasing sequence of finite partitions $\{\mathcal{F}_t\}_{t=0}^{\infty}$ of $\mathcal{S}$. A state $s \in \mathcal{S}$ specifies a complete history of the environment from date 0 to the infinite future. The partition $\mathcal{F}_t$ specifies sets of states that can be verified by the information available at date t. An element $s^t \in \mathcal{F}_t$ is called a date-t event. We take $\mathcal{F}_0 = \mathcal{S}$ so that there is no uncertainty at date 0.

This description of the uncertain environment can be interpreted as an event tree. An event $s^t \in \mathcal{F}_t$ at date t identifies a node of the event tree. The unique predecessor of s^t is a date-$(t-1)$ event $s^t_- \in \mathcal{F}_{t-1}$ such that $s^t \subset s^t_-$. Immediate successors of s^t are date-$(t+1)$ events $\{s^t_+\}$ such that $s^t_+ \in \mathcal{F}_{t+1}$ and $s^t_+ \subset s^t$. The unique date 0 event s^0 is the root node of the event tree.

The set of all events at all dates is denoted by $\mathcal{E}$. The set of successor events of s^t is denoted by $\mathcal{E}^+(s^t)$, the set of all date-τ successor events of s^t is denoted by $\mathcal{F}_\tau(s^t)$. We also write $\mathcal{E}(s^t) \equiv s^t \cup \mathcal{E}^+(s^t)$.

3. Arrow-Debreu Equilibrium

There is a single consumption good at every date, in every event (see Section 6 for a discussion of an extension to multiple goods). A consumption plan is a scalar-valued process adapted to $\{\mathcal{F}_t\}_{t=0}^{\infty}$. Consumption plans are restricted to a a linear space of adapted processes C. Our primary choice of the consumption space C is the space of all adapted processes (which can be identified with $\mathcal{R}^\infty$). The cone of nonnegative processes in C is denoted by C_+; a typical element of C is denoted by $c = \{c(s^t)\}$.

There are I consumers. Each consumer i has the consumption set C_+, a strictly monotone and complete preference $\preceq^i$ on C_+, and an intial endowment $\omega^i \in C_+$.

The standard notion of an Arrow-Debreu general equilibrium is extended to our setting with infinitely many dates as follows: Prices are described by linear functional P which is positive and well defined (and finite) on each consumer's initial endowment. We call such functional a *pricing functional*. It follows that a pricing functional is well defined on the aggregate endowment $\bar{\omega}$ and, since it is positive, also on each consumption plan satisfying $0 \leq c \leq \bar{\omega}$. It may or may not be well defined on the entire space C.

The price of one unit of consumption in event s^t under pricing functional P is $p(s^t) \equiv P(e(s^t))$, where $e(s^t)$ denotes the consumption plan equal to 1 in event s^t at date t and zero in all other events and all other dates. We define the implicit price of consumption plan c in event s^t as $P_{s^t}(c) \equiv P(c|s^t)/p(s^t)$, where $c|s^t$ is a consumption plan equal to c in s^t and all its successor events, and zero in all other events. A pricing functional P is countably additive, if and only if $P(c) = \sum_{\mathcal{E}} p(s^t)c(s^t)$ for every c for which $P(c)$ is well defined. If P is countably additive then $P_{s^t}(c) = [1/p(s^t)] \sum_{\mathcal{E}(s^t)} p(s^t)c(s^t)$.

An *Arrow-Debreu equilibrium* is a pricing functional P and a consumption allocation $\{c^i\}$ such that c^i maximizes consumer i's preference $\preceq^i$ subject to $P(c) \leq P(\omega^i)$ and $c \in C_+$, and markets clear, that is $\sum_i c^i = \sum_i \omega^i$.

We will also need the notion of an equilibrium with wealth transfers. For given wealth transfers ϵ^i such that $\sum_i \epsilon^i = 0$, a pricing functional P and a consumption allocation $\{c^i\}$ are an *Arrow-Debreu equilibrium with wealth transfers* if c^i maximizes consumer i's preference $\preceq^i$ subject to $P(c) \leq P(\omega^i) + \epsilon^i$ and $c \in C_+$, and markets clear.

3. Sequential Equilibrium and Price Bubbles

The notion of sequential equilibrium applies to markets in which trade takes place at every date. We consider J infinitely-lived securities traded at every date. Each security j is specified by a dividend process d_j which is adapted to $\{\mathcal{F}_t\}_{t=0}^{\infty}$ and nonnegative. For at least one security j the dividend process d_j is not eventually zero, that is, for each s^t there exists a succesor event s^τ such that $d_j(s^\tau) > 0$ The ex-dividend price of security j in event s^t is denoted by $q_j(s^t)$ and q_j is the price process of security j. A portfolio strategy θ specifies a portfolio of J securities $\theta(s^t)$ held after trade in event s^t for each s^t. Thus θ is a $I\!R^J$-valued process adapted to $\{\mathcal{F}_t\}_{t=0}^{\infty}$.

Each consumer i has an initial portfolio $\bar{\theta}^i$ at date 0. The supply of securities is therefore $\bar{\theta} = \sum_i \bar{\theta}^i$. We assume that $\bar{\theta} \geq 0$. When comparing sequential equilibria with Arrow-Debreu equilibria we want to allow for nonzero supply of the securities and at the same time have the total endowment of goods unchanged. Therefore we specify consumption endowments in sequential markets by $y^i \in C$ and assume that

$$\omega^i = y^i + \bar{\theta}^i d. \tag{1}$$

Consumers face feasibility constraints when choosing their portfolio strategies. Such constraints are necessary to prevent consumers from using Ponzi , see Huang and Werner (1999). In the definition of sequential equilibrium below the set of feasible portfolio strategies of consumer i is Θ^i. Specific feasibility constraints will be introduced in later in this section.

A *sequential equilibrium* is a price process q and consumption-portfolio allocation $\{c^i, \theta^i\}$ such that:

(i) for each i, consumtion plan c^i and portfolio strategy θ^i maximize $\preceq^i$ subject to

$$c(s^0) + q(s^0)\theta(s^0) \leq y^i(s^0) + q(s^0)\bar{\theta}^i,$$
$$c(s^t) + q(s^t)\theta(s^t) \leq y^i(s^t) + [q(s^t) + d(s^t)]\theta(s^t_-) \quad \forall s^t \neq s^0,$$
$$c \in C_+,\ \theta \in \Theta^i;$$

(ii) markets clear, that is

$$\sum_h c^i(s^t) = y(s^t) + \bar{\theta} d(s^t), \quad \sum_h \theta^i(s^t) = \bar{\theta}, \qquad \forall s^t$$

Security markets are *one-period complete* in event s^t at prices q if the one-period payoff matrix $\{q(s^{t+1})+d(s^{t+1})\}_{s^{t+1}\in\{s^t_+\}}$has rank equal to the number of immediate successors of s^t. Security markets are *complete* at q if they are one-period complete at every event. Of course, for markets to be complete it is neccessary that the number of securities is at least as large as the number of immediate successors of any event.

If markets are complete, then for each event s^t there exists a portfolio strategy that has payoff equal to one in s^t, zero at every other event and involves no holding after date t. The date-0 price of that portfolio strategy is the *event price* (present value) of one unit of consumption in event s^t, denoted $p(s^t)$.

The *present value* at s^t of security j is defined using event prices as

$$\frac{1}{p(s^t)} \sum_{s^\tau \in \mathcal{E}^+_{s^t}} p(s^\tau) d_j(s^\tau), \tag{2}$$

where we assumed that $p(s^t) \neq 0$ for every s^t. To see that this infinite sum is well defined, we first observe the following relation between event prices and security prices

$$p(s^t) q_j(s^t) = \sum_{s^{t+1}\in\{s^t_+\}} p(s^{t+1}) \left[q_j(s^{t+1}) + d_j(s^{t+1}) \right] \quad \forall s^t, j. \tag{3}$$

Using (3) recursively we obtain

$$q_j(s^t) = \Big(\sum_{\tau=t+1}^{r} \sum_{s^\tau \in \mathcal{F}_\tau(s^t)} \frac{p(s^\tau)}{p(s^t)} d_j(s^\tau) \Big) + \sum_{s^r \in \mathcal{F}_r(s^t)} \frac{p(s^r)}{p(s^t)} q_j(s^r) \tag{4}$$

for each s^t, and for any $r > t$. Since the price $q_j(s^t)$ is nonnegative, (4) implies that

$$q_j(s^t) \geq \sum_{\tau=t+1}^{r} \sum_{s^\tau \in \mathcal{F}_\tau(s^t)} \frac{p(s^\tau)}{p(s^t)} d_j(s^\tau) \tag{5}$$

for every s^t and $r > t$. Since dividends $d_j(s^\tau)$ are nonnegative, we can take the limit on the right hand side of (5) as r goes to infinity and we obtain that the present value (2) is well defined and does not exceed the the price of the security.

The positive difference between the price and the present value of security j is the *price bubble* on that security. We denote the price bubble by $\sigma_j(s^t)$ so that

$$\sigma_j(s^t) \equiv q_j(s^t) - \frac{1}{p(s^t)} \sum_{s^\tau \in \mathcal{E}_{s^t} \setminus \{s^t\}} p(s^\tau) d_j(s^\tau). \tag{6}$$

We have that $\sigma_j(s^t) \geq 0$.

For use later, we note that (4) and (6) imply that

$$\sigma_j(s^t) = \sum_{s^{t+1} \in s^t_+} \frac{p(s^{t+1})}{p(s^t)} \sigma_j(s^{t+1}) \tag{7}$$

for each s^t.

Whether nonzero price bubbles can exist in a sequential equilibrium depends crucially on the form of portfolio feasibility constraints (see Huang and Werner (1999)). There are two feasibility constraints that are important for our implementation results. The first applies to complete security markets. It prohibits a consumer from borrowing more than the present value of his future endowment, that is

$$q(s^t)\theta^i(s^t) \geq -\frac{1}{p(s^t)} \sum_{s^\tau \in \mathcal{E}^+(s^t)} p(s^\tau) y^i(s^\tau) \qquad \forall s^t. \tag{8}$$

We refere to constraint (8) as the *wealth constraint.*

If the supply of securities is strictly positive, that is $\bar{\theta} >> 0$, then there cannot be nonzero price bubbles in a sequential equilibrium (Santos and Woodford (1997)). However, if some securities are in zero supply, then nonzero price bubbles are possible.

To introduce the second portfolio feasibility constraint we first define a normalization of security prices. Whenever security price vector $q(s^t)$ is positive and nonzero we can define the *normalized security price vector* $\hat{q}(s^t)$ by

$$\hat{q}(s^t) \equiv \frac{q(s^t)}{\sum_j q_j(s^t)} \tag{9}$$

Under normalization (9) security prices are measured relative to the price of portfolio consisting of one share of each security. We could have used any other portfolio with strictly positive weights as a reference portfolio in the definition of price normalization.

Portfolio strategy θ satisfies the *bounded borrowing constraint* if

$$\inf_{s^t \in \mathcal{E}} \hat{q}(s^t)\theta(s^t) > -\infty. \tag{10}$$

We refer to a sequential equilibrium in which each consumer's set of feasible portfolio strategies consists of all portfolio strategies satisfying (10) as a sequential equilibrium with *bounded borrowing.*

There cannot be a nonzero price bubble in sequential equilibrium with bounded borrowing.

Theorem 2.1. *If q is a sequential equilibrium price process with bounded borrowing and if security markets are complete at q, then $\sigma_j(s^t) = 0$ for every j and every s^t.*

Proof: We prove that $\sigma_j(s^0) = 0$ for every j. The same argument will apply to any event s^t for $t > 0$.

Since security markets are complete at q, for each event s^t and each security j there exists a portfolio $\eta^j(s^t)$ such that

$$[q(s^{t+1}) + d(s^{t+1})]\eta^j(s^t) = \frac{1}{p(s^{t+1})} \sum_{s^r \in \mathcal{E}(s^(t+1)} p(s^r)d_j(s^r) \qquad \forall s^{t+1} \in s^t_+, \tag{11}$$

If we multiply both sides of (11) by $p(s^{t+1})$, sum over all $s^{t+1} \in s^t_+$ and use (), then we obtain

$$q(s^t)\eta^j(s^t) = \frac{1}{p(s^t)} \sum_{s^r \in \mathcal{E}(s^(t)} p(s^r)d_j(s^r). \tag{12}$$

Equations (11) and (12) imply that

$$[q(s^t) + d(s^t)]\eta^{jt}(s^t_-) - q(s^r)\eta^{jt}(s^t) = d_j(s^t) \qquad \forall s^t.$$

Thus η^j is a portfolio strategy with payoff that replicates the dividend d_j of security j and with date-0 price equal to the present value of d_j at date 0.

Suppose that $\sigma_j(s^0) > 0$ for some j. Consider a portfolio strategy θ^j that results from holding η^j and selling one share of security j at date 0. Thus $\theta^j_j(s^t) = \eta^j_j(s^t) - 1$ and $\theta^j_k(s^t) = \eta^j_k(s^t)$ for $k \neq j$ and for every s^t. We have

$$[q(s^t) + d(s^t)]\theta^j(s^t_-) - q(s^t)\theta^j(s^t) = 0 \qquad \forall s^t \neq s^0.$$

Using (12) and (6) we also have

$$q(s^t)\theta^j(s^t) = -\sigma_j(s^t) \qquad \forall s^t. \tag{13}$$

Further, since

$$\frac{-\sigma_j(s^t)}{\sum_j q_j(s^t)} \geq -1 \qquad \forall s^t,$$

strategy θ^j is feasible under the bounded borrowing constraint.

Let θ^i be consumer i equilibrium portfolio strategy. Consider portfolio strategy $\theta^j + \theta^j$. This strategy is budget feasible, satisfies the bounded borrowing constraint and generates higher consumption at date 0 without decreasing consumption in any other event. This is contradiction to the assumption that c^i and θ^i are equilibrium consumption and portfolio choices. Therefore $\sigma(s^t) = 0$ for every s^t. $\square$

5. Implementation.

Theorem 5.1. *If consumption allocation $\{c^i\}$ and a countably additive pricing functional P are an Arrow-Debreu equilibrium such that security markets are complete at prices q given by*

$$q_j(s^t) = P_{s^t}(d_j), \tag{14}$$

then there exists a portfolio allocation $\{\theta^i\}$ such that q and the allocation $\{c^i, \theta^i\}$ are a sequential equilibrium under the wealth constraint.

We defer the proof of Theorem 5.1 until later in this section when we state and prove a more general theorem.

Theorem 5.1 says that an Arrow-Debreu equilibrium in which the implicit security prices with zero price bubbles make the security markets complete can be implemented as a sequential equilibrium under the wealth constraint. Our next result says that the implementation is exact when sequential equilibria with zero price bubbles are considered.

Theorem 5.2 *If security prices q and consumption-portfolio allocation $\{c^i, \theta^i\}$ are a sequential equilibrium under the wealth constraint, security markets are complete at q and price bubbles are zero, then consumption allocation $\{c^i\}$ and the pricing functional P given by $P(c) = \sum_{s^t \in \mathcal{E}} p(s^t)c(s^t)$, where $p(s^t)$ is the event price of s^t, are an Arrow-Debreu equilibrium.*

Again, the proof of Theorem 5.2 is deferred until later in this section.

If the supply of the securities is strictly positive, then, as mentioned in Section 4, there are cannot be a nonzero price bubble in a sequential equilibrium. Then Theorems 5.1 and 5.2 provide a one-to-one relation between Arrow-Debreu equilibria and sequential equilibria. However, if the supply of some securities is zero, sequential equilibria with nonzero price bubbles are possible. These equilibria do not correspond to Arrow-Debreu equilibria. We prove now that the equilibria with nonzero price bubbles are Arrow-Debreu equilibria with wealth transfers.

Theorem 5.3. *Let $\sigma(s^0) \in I\!R^J_+$ and satisfy $\sigma(s^0)\bar{\theta} = 0$. If consumption allocation $\{c^i\}$ and countably additive pricing functional P are an Arrow-Debreu equilibrium with wealth transfers $\{\sigma(s^0)\bar{\theta}^i\}$ and security markets are complete at prices q given by*

$$q_j(s^t) = \sigma_j(s^t) + P_{s^t}(d_j) \tag{15}$$

for some $\sigma(s^t) \in I\!R^J_+$ satisfying (7), then there exists a portfolio allocation $\{\theta^i\}$ such that q and allocation $\{c^i, \theta^i\}$ are a sequential equilibrium under the wealth constraint.

Conversely

Theorem 5.4. *If security prices q and consumption-portfolio allocation $\{c^i, \theta^i\}$ are a sequential equilibrium under the wealth constraint and if security markets are complete at q, then consumption allocation $\{c^i\}$ and the pricing functional P given by $P(c) = \sum_{s^t \in \mathcal{E}} p(s^t)c(s^t)$, where $p(s^t)$ is the event price of s^t, are an Arrow-Debreu equilibrium with with wealth transfers $\{\sigma(s^0)\bar{\theta}^i\}$, where $\sigma(s^0)$ is the vector of price bubbles at date 0.*

Theorems 5.3 and 5.4 indicate that there is a multiplicity of sequential equilibria under the wealth constraint when some securities are in zero supply. Unless the initial portfolios of securities are all zero, the multiplicity is reflected in different consumption allocations.

We turn now our attention to sequential equilibria under the bounded borrowing constraint. It follows from Theorem 2.1 that there cannot be a nonzero price bubble in a sequential equilibrium with bounded borrowing. We have

Theorem 5.5. *If consumption allocation $\{c^i\}$ and a countably additive pricing functional P are an Arrow-Debreu equilibrium such that security markets are complete at prices q given by*

$$q_j(s^t) = P_{s^t}(d_j), \tag{16}$$

and there exists a portfolio strategy η which is bounded and

$$\bar{y}(s^t) \leq [q(s^t) + d(s^t)]\eta(s^t_-) - q(s^t)\eta(s^t) \quad \forall s^t \tag{17}$$

then there exists a portfolio allocation $\{\theta^i\}$ such that q and the allocation $\{c^i, \theta^i\}$ are a sequential equilibrium with bounded borrowing.

Conversely,

Theorem 5.6. *If security prices q and consumption-portfolio allocation $\{c^i, \theta^i\}$ are a sequential equilibrium with bounded borrowing, security markets are complete at q, then consumption allocation $\{c^i\}$ and the pricing functional P given by $P(c) = \sum_{s^t \in \mathcal{E}} p(s^t)c(s^t)$, where $p(s^t)$ is the event price of s^t, are an Arrow-Debreu equilibrium.*

A sufficient condition for there being a bounded portfolio strategy η satisfying (17) is that $\bar{y}$ is eventually bounded relative to $\sum_j d_j$, that is, $\bar{y}(s^t) \leq \gamma^i \sum_j d_j(s^t)$ for all s^t where $t \geq \tau$, for some $\gamma^i > 0,\ \tau > 0$.

PROOFS:

Proof of 5.4: It suffices to show that for each consumer i the sets of budget feasible consumption plans are the same in sequential markets with the wealth constraint under security prices q and in Arrow-Debreu markets under the specified countably additive pricing functional P and wealth transfers $\{\sigma(s^0)\bar{\theta}^i\}$.

If c^i is budget feasible in sequential markets with the wealth constraint under q, then there exists a portfolio strategy θ^i such that

$$
\begin{aligned}
& c^i(s^0) + q(s^0)\theta^i(s^0) \leq y^i(s^0) + q(s^0)\bar{\theta}^i, \\
& c^i(s^t) + q(s^t)\theta^i(s^t) \leq y^i(s^t) + [q(s^t) + d(s^t)]\theta^i(s^t_-) \qquad \forall s^t \neq s^0, \\
& q(s^t)\theta^i(s^t) \geq -[1/p(s^t)] \sum_{s^\tau \in \mathcal{E}^+(s^t)} p(s^\tau) y^i(s^\tau) \qquad \forall s^t.
\end{aligned}
$$

Multiplying both sides of the budget constraints by $p(s^t)$ and summing over all s^t for t ranging from 0 to some arbitrary τ, and using (3), we obtain

$$
\sum_{t=0}^{\tau} \sum_{s^t \in \mathcal{F}_t} p(s^t) c^i(s^t) + \sum_{s^\tau \in \mathcal{F}_\tau} p(s^\tau) q(s^\tau) \theta^i(s^\tau) \leq \sum_{t=0}^{\tau} \sum_{s^t \in \mathcal{F}_t} p(s^t) y^i(s^t) + q(s^0)\bar{\theta}^i. \quad (18)
$$

Adding $\sum_{t=\tau+1}^{\infty} \sum_{s^t \in \mathcal{F}_t} p(s^t) y^i(s^t)$ to both sides of (18) and using the wealth constraint lead to

$$
\begin{aligned}
& \sum_{s^t \in \mathcal{E}} p(s^t) y^i(s^t) + q(s^0)\bar{\theta}^i \\
\geq \quad & \sum_{t=0}^{\tau} \sum_{s^t \in \mathcal{F}_t} p(s^t) c^i(s^t) + \sum_{s^\tau \in \mathcal{F}_\tau} \left[p(s^\tau) q(s^\tau) \theta^i(s^\tau) + \sum_{s^t \in \mathcal{E}^+(s^\tau)} p(s^t) y^i(s^t) \right] \\
\geq \quad & \sum_{t=0}^{\tau} \sum_{s^t \in \mathcal{F}_t} p(s^t) c^i(s^t). \qquad (19)
\end{aligned}
$$

Taking limits in (19) as τ goes to infinity results in

$$\sum_{s^t \in \mathcal{E}} p(s^t) c^i(s^t) \le \sum_{s^t \in \mathcal{E}} p(s^t) y^i(s^t) + q(s^0)\bar{\theta}^i. \tag{20}$$

Since $\sum_{s^t \in \mathcal{E}} p(s^t) y^i(s^t)$ is finite in sequential equilibrium under the wealth constrain, $P(y^i)$ is finite in Arrow-Debreu markets under the countably additive pricing functional P. In light of endowment relation (1) and inequality (20), $P(\omega^i)$ and $P(c^i)$ are also finite under P. Using (6) for s^0 and (1), (20) simplifies to $P(c^i) \le P(\omega^i) + \sigma(s^0)\bar{\theta}^i$. Thus c^i satisfies the Arrow-Debreu budget constraint under P with wealth transfers $\{\sigma(s^0)\bar{\theta}^i\}$.

Consider now a consumption plan c^i that is budget feasible in Arrow-Debreu markets under P with wealth transfers $\{\sigma(s^0)\bar{\theta}^i\}$, that is

$$P(c^i) \le P(\omega^i) + \sigma(s^0)\bar{\theta}^i. \tag{21}$$

That $P(\omega^i)$ is finite and (21) imply that $P(c^i)$ is finite. Since security markets are complete at q and $P(c^i)$ and $P(y^i)$ are finite, for each s^t there exists portfolio $\theta^i(s^t)$ such that

$$[q(s^{t+1}) + d(s^{t+1})]\theta^i(s^t) = P_{s^t}(c^i) - P_{s^t}(y^i) \qquad \forall s^{t+1} \in s^t_+, \tag{22}$$

Since P is countably additive, equation (22) can be written as

$$[q(s^{t+1}) + d(s^{t+1})]\theta^i(s^t) = \sum_{s^\tau \in \mathcal{E}(s^{t+1})} \frac{p(s^\tau)}{p(s^{t+1})}(c^i(s^\tau) - y^i(s^\tau)) \qquad \forall s^{t+1} \in s^t_+. \tag{23}$$

Multiplying both sides of (23) by $p(s^{t+1})$, summing over all $s^{t+1} \in s^t_+$, and using (3), we obtain

$$q(s^t)\theta^i(s^t) = \sum_{s^\tau \in \mathcal{E}^+(s^t)} \frac{p(s^\tau)}{p(s^t)}(c^i(s^\tau) - y^i(s^\tau)) \qquad \forall s^t. \tag{24}$$

It follows from (23) and (24) that

$$c^i(s^t) + q(s^t)\theta^i(s^t) = y^i(s^t) + [q(s^t) + d(s^t)]\theta^i(s^t_-) \qquad \forall s^t \neq s^0. \tag{25}$$

Arrow-Debreu budget constraint (21), equation (6), and endowment relation (1) imply that $P(c^i) \leq P(\omega^i) + \sigma(s^0)\bar{\theta}^i = P(y^i) + q(s^0)\bar{\theta}^i$, which together with (24) for s^0 lead to

$$c^i(s^0) + q(s^0)\theta^i(s^0) \leq y^i(s^0) + q(s^0)\bar{\theta}^i. \tag{26}$$

(25) and (26) imply that consumption-portfolio plan (c^i, θ^i) satisfies the sequential budget constraints under q. Moreover, since $c^i \geq 0$, equality (24) implies that θ^i satisfies the wealth constraint. This completes the proof.□

Proof of 5.3: Similarly as in the proof of Theorem 5.4 we can show that for each consumer i the sets of budget feasible consumption plans are the same in Arrow-Debreu markets under countably additive pricing functional P with wealth transfers $\{\sigma(s^0)\bar{\theta}^i\}$ and in sequential markets under the wealth constraint and the security prices q specified by (15). Specifically, since sequential markets are complete at q and $P(\omega^i)$ and $P(y^i)$ are finite in Arrow-Debreu equilibrium, for each budget feasible consumption plan c^i in Arrow-Debreu markets $P(c^i)$ is finite and there exists a portfolio plan θ^i such that (22)-(23) hold. Plan (c^i, θ^i) satisfies the sequential budget constraints and θ^i satisfies the wealth constraint under q.

It is worth noting that for each consumer i there generally exist more than one such portfolio plan θ^i if the number of securities is larger than the number of immediate successors of some event. For each s^t, let $J(s^t)$ be a collection of as many securities with linearly independent one-period payoffs as the immediate successors of s^t. We consider such portfolio plans $\{\theta^i\}$ for which in each event s^t

$$\sum_i \theta^i_j(s^t) = \bar{\theta}_j \qquad \forall j \notin J(s^t). \tag{27}$$

We now show that these portfolio plans clear security markets if $\{c^i\}$ are Arrow-Debreu equilibrium consumption allocations under P. To proceed, summing (23) over all i, and using the market clearing condition for consumption $\sum_i c^i = \sum_i \omega^i$ and the endowment relation (1), we obtain

$$[q(s^t) + d(s^t)] \sum_i \theta^i(s^t_-) = \sum_{s^\tau \in \mathcal{E}(s^t)} \frac{p(s^\tau)}{p(s^t)} d(s^\tau)\bar{\theta} \qquad \forall s^t \neq s^0. \tag{28}$$

Using (15) and the fact that P is countably additive, we can rewrite (28) as

$$[q(s^t) + d(s^t)] \left[\sum_i \theta^i(s^t_-) - \bar{\theta} \right] = -\sigma(s^t)\bar{\theta} \qquad \forall s^t \neq s^0. \tag{29}$$

Since preferences are nonsatiated, the Arrow-Debreu budget constraint holds with equality in equilibrium, that is, $P(c^i) = P(\omega^i) + \sigma(s^0)\bar{\theta}^i(s^0)$. This observation combined with the market clearing condition for consumption implies that

$$\sigma(s^0)\bar{\theta} = 0. \tag{30}$$

In light of (7) and (30), we have

$$\sum_{s^t \in \mathcal{F}_t} \frac{p(s^t)}{p(s^0)} \sigma(s^t)\bar{\theta} = \sigma(s^0)\bar{\theta} = 0 \qquad \forall t \geq 0. \tag{31}$$

Since $\bar{\theta} \in I\!R^J_+$ and $\sigma(s^t) \in I\!R^J_+$ for all s^t, (31) implies that

$$\sigma(s^t)\bar{\theta} = 0 \qquad \forall s^t. \tag{32}$$

Given (32), equation (29) reduces to

$$[q(s^t) + d(s^t)] \left[\sum_i \theta^i(s^t_-) - \bar{\theta} \right] = 0 \qquad \forall s^t \neq s^0. \tag{33}$$

Since sequential markets are complete at q, equations (27) and (33) imply that $\sum_i \theta^i(s^t) = \bar{\theta}$ for all s^t, which means that $\{\theta^i\}$ clear security markets. This completes the proof.□

Proof of 5.6: It suffices to show that for each consumer i the sequential budget set under the bounded borrowing constraint at security prices q coincides with the set of budget feasible consumptions in Arrow-Debreu markets at the specified countably additive pricing functional P.

Let c^i be budget feasible for consumer i in sequential markets at q with bounded borrowing. Then there exists a portfolio strategy θ^i such that

$$c^i(s^0) + q(s^0)\theta^i(s^0) \leq y^i(s^0) + q(s^0)\bar{\theta}^i,$$
$$c^i(s^t) + q(s^t)\theta^i(s^t) \leq y^i(s^t) + [q(s^t) + d(s^t)]\theta^i(s^t_-) \qquad \forall s^t \neq s^0,$$
$$\inf_{s^t \in \mathcal{E}} \hat{q}(s^t)\theta^i(s^t) > -\infty.$$

These budget constraints imply (18).

We claim that

$$\liminf_{\tau\to\infty} \sum_{s^\tau\in\mathcal{F}_\tau} p(s^\tau)q(s^\tau)\theta^i(s^\tau) \geq 0. \tag{34}$$

To prove (34) note that since θ^i involves bounded borrowing at q, (9) and (10) imply that $q(s^\tau)\theta^i(s^\tau) \geq B\sum_j q_j(s^\tau)$ for all s^τ, and for some B. It follows that

$$\sum_{s^\tau\in\mathcal{F}_\tau} p(s^\tau)q(s^\tau)\theta^i(s^\tau) \geq B\sum_j \sum_{s^\tau\in\mathcal{F}_\tau} p(s^\tau)q_j(s^\tau) \qquad \forall s^\tau. \tag{35}$$

Taking $\tau\to\infty$ in (35) and using (**??**) result in

$$\liminf_{\tau\to\infty} \sum_{s^\tau\in\mathcal{F}_\tau} p(s^\tau)q(s^\tau)\theta^i(s^\tau) \geq B\sum_j \sigma_j(s^0).$$

By Theorem 2.1 there is no price bubble in sequential equilibrium with bounded borrowing; particularly, $\sigma_j(s^0)=0$ for every j. The desired result then follows.

Taking limits in (18) as τ goes to infinity leads to

$$\sum_{s^t\in\mathcal{E}} p(s^t)c^i(s^t) + \liminf_{\tau\to\infty} \sum_{s^\tau\in\mathcal{F}_\tau} p(s^\tau)q(s^\tau)\theta^i(s^\tau) \leq \sum_{s^t\in\mathcal{E}} p(s^t)y^i(s^t) + q(s^0)\bar{\theta}^i. \tag{36}$$

Substituting (34) into (36) yields

$$\sum_{s^t\in\mathcal{E}} p(s^t)c^i(s^t) \leq \sum_{s^t\in\mathcal{E}} p(s^t)y^i(s^t) + q(s^0)\bar{\theta}^i. \tag{37}$$

Note that $\sum_{s^t\in\mathcal{E}} p(s^t)y^i(s^t)$ is finite in sequential equilibrium since there exists a portfolio plan η which is bounded and satisfies (17). Thus $P(y^i)$ is finite in Arrow-Debreu markets under the countably additive pricing functional P. By virtue of (1) and (37), $P(\omega^i)$ and $P(c^i)$ are also finite under P. The non-existence of price bubble and (6) imply that

$$q_j(s^t) = [1/p(s^t)] \sum_{s^\tau\in\mathcal{E}^+(s^t)} p(s^\tau)d_j(s^\tau) \qquad \forall s^t \quad \forall j. \tag{38}$$

In light of (1) and (38), inequality (37) can be written as $P(c^i)\leq P(\omega^i)$. Thus c^i satisfies the Arrow-Debreu budget constraint under P.

Suppose now that a consumption plan c^i is budget feasible in Arrow-Debreu markets under P, that is

$$P(c^i) \leq P(\omega^i). \tag{39}$$

Since $P(\omega^i)$ is finite, (39) implies that $P(c^i)$ is finite. Since security markets are complete at q and $P(c^i)$ and $P(y^i)$ are finite, there exists a portfolio plan θ^i such that (22)-(25) hold. Equation (38) also holds due to Theorem 2.1. (1), (38) and (39) imply that $P(c^i) \leq P(\omega^i) = P(y^i) + q(s^0)\bar{\theta}^i$, which together with (24) for s^0 give (26). (25) and (26) say that consumption-portfolio plan (c^i, θ^i) satisfies the sequential budget constraints under q.

It remains to show that θ^i satisfies the bounded brrowing constraint. Since $c^i \geq 0$, (24) implies that

$$q(s^t)\theta^i(s^t) \geq -[1/p(s^t)] \sum_{s^\tau \in \mathcal{E}^+(s^t)} p(s^\tau) y^i(s^\tau) \qquad \forall s^t. \tag{40}$$

Since there exists a portfolio plan η which is bounded and satisfies (17), y^i is eventually bounded relative to $\sum_j d_j$. Therefore, there exists some γ such that

$$q(s^t)\theta^i(s^t) \geq -[\gamma/p(s^t)] \sum_{s^\tau \in \mathcal{E}^+(s^t)} p(s^\tau) \sum_j d_j(s^\tau) \qquad \forall s^t. \tag{41}$$

Substituting (38) into (41) yields

$$q(s^t)\theta^i(s^t) \geq -\gamma \sum_j q_j(s^t) \qquad \forall s^t. \tag{42}$$

It follows from (9) and (42) that $\hat{q}(s^t)\theta^i(s^t) \geq -\gamma$ for all s^t. Thus θ^i satisfies the bounded borrowing constraint under q.$\square$

Proof of 5.5: Similarly as in the proof of Theorem 5.6 we can show that for each consumer i the Arrow-Debreu budget set under countably additive pricing functional P coincides with the set of budget feasible consumptions in sequential markets at the security prices q specified by (16) under the bounded borrowing constraint. Particularly, since sequential markets are complete at q and $P(\omega^i)$ and

$P(y^i)$ are finite in Arrow-Debreu equilibrium, for each budget feasible consumption plan c^i in Arrow-Debreu markets $P(c^i)$ is finite and there exists a portfolio plan θ^i such that (22)-(25) hold. The Arrow-Debreu budget constraint $P(c^i) \leq P(\omega^i)$, price relation (16), and endowment relation (1) imply that $P(c^i) \leq P(\omega^i) = P(y^i) + q(s^0)\bar{\theta}^i$. This together with (24) for s^0 and the fact that P is countably additive result in (26). (25) and (26) say that plan (c^i, θ^i) satisfies the sequential budget constraints under q. To show that θ^i satisfies the bounded borrowing constraint, note that (40)-(41) hold for the same reason stated in the proof of Theorem 5.6. Since P is countably additive, substituting (16) into (41) yields (42). The said result then follows.

Among all such portfolio plans $\{\theta^i\}$ consider those that satisfy (27). We claim that these plans clear security markets. To prove our claim, summing (23) over all i, and using the market clearing condition for consumption $\sum_i c^i = \sum_i \omega^i$ and endowment relation (1), we obtain (28). Substituting (16) into (28) and using the fact that P is countably additive lead to (33). Since security markets are complete at q, equations (27) and (33) imply that $\sum_i \theta^i(s^t) = \bar{\theta}$ for all s^t. Therefore, $\{\theta^i\}$ clear security markets. This completes the proof.$\square$

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